

THE BARTNIK-BRAY OUTER MASS OF SMALL METRIC SPHERES IN TIME-SYMMETRIC 3-SLICES

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ABSTRACT. Given a sphere with Bartnik data close to that of a round sphere in Euclidean 3-space, we compute its Bartnik-Bray outer mass to first order in the data's deviation from the standard sphere. The Hawking mass gives a well-known lower bound, and an upper bound is obtained by estimating the mass of a static vacuum extension. As an application we confirm that in a time-symmetric slice concentric geodesic balls shrinking to a point have mass-to-volume ratio converging to the energy density at their center, in accord with physical expectation and the behavior of other quasilocal masses. For balls shrinking to a flat point we can also compute the outer mass to fifth order in the radius—the term is proportional to the Laplacian of the scalar curvature at the center—but our estimate is not refined enough to identify this term at a point which is merely scalar flat. In particular it cannot discern gravitational contributions to the mass.

1. INTRODUCTION

When studying the properties of a given definition for quasilocal mass, it is natural to investigate the asymptotic behavior of the mass on large and small spheres (or the regions they bound) so as to assess its relation to the canonical measures of mass in those regimes, namely (for regions in time-symmetric slices) the ADM energy in the large and the matter-field energy density in the small. We have for example [16], [10], [28], [15], and [11], concerning a variety of quasilocal masses on small spheres. For the Bartnik mass ([6], [7], [8]) in particular, while Huisken and Ilmanen have addressed the large-sphere limit in [17], there do not appear to be any asymptotic estimates for small spheres in the literature, and indeed the question of the (suitably scaled) small-sphere limit has been raised explicitly by Szabados in the review [26]. Here we give a partial answer, substituting Bray's outer-minimizing condition ([9]) for Bartnik's no-horizon condition and allowing extensions which are not smooth across the boundary. In fact the precise quasilocal mass we consider is essentially one studied by Miao (whose sign convention for mean curvature we caution is opposite ours below) in Section 3.3 of [21].

Specifically, let \mathcal{PM} denote the set of complete Riemannian metrics on $M := \{|x| \geq 1\} \subset \mathbb{R}^3$ having nonnegative scalar curvature and whose Cartesian components g_{ij} are C^2 up to the boundary and satisfy the decay requirements

$$(1.1) \quad |g_{ij}(x) - \delta_{ij}| + |x| |g_{ij,k}(x)| + |x|^2 |g_{ij,k\ell}(x)| = O(|x|^{-1}).$$

Given any metric g on M or on the closed unit ball B and writing $\iota : S^2 \rightarrow \mathbb{R}^3$ for the inclusion map of the unit sphere $S^2 = \partial M$ in \mathbb{R}^3 , we adopt the sign convention that the mean curvature $\mathcal{H}_\iota[g]$ induced by g on ∂M is the divergence of the unit normal directed away from the origin (so that $\mathcal{H}_\iota[\delta] = -2$ for the standard Euclidean metric δ). We recall that ∂M is said to be *outer-minimizing* in (M, g) if its area is no greater than that of any enclosing surface.

Now given a C^2 metric γ and a C^1 function H on S^2 , we define the class

$$(1.2) \quad \mathcal{PM}[\gamma, H] := \{g \in \mathcal{PM} \mid \iota^*g = \gamma \text{ and } \mathcal{H}_\iota[g] \geq H\}$$

and the *Bartnik-Bray outer mass* of (S^2, γ, H)

$$(1.3) \quad m_{out}[\gamma, H] := \inf\{m_{ADM}[g] \mid g \in \mathcal{PM}[\gamma, H] \text{ and } \partial M \text{ is outer-minimizing in } (M, g)\},$$

where

$$(1.4) \quad m_{ADM}[g] := \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{|x|=r} \left(g_{ik}^{,k} - g^k_{k,i} \right) \frac{x^i}{|x|}$$

is the ADM mass ([4],[5],[13],[23]) of (M, g) . Given a Riemannian 3-manifold-with-boundary (N, g_0) diffeomorphic to the closed unit ball B , we also define the Bartnik-Bray outer mass of (N, g_0)

$$(1.5) \quad m_{out}[N, g_0] := m_{out}[S^2, \iota^* \phi^* g_0, \mathcal{H}_\iota[\phi^* g_0]]$$

with $\phi : B \rightarrow N$ any diffeomorphism.

As some motivation for the above definitions we mention that the imposed equality on the induced metric and inequality on the mean curvature ensure nonnegative distributional scalar curvature of the extension of g_0 by g as well as the applicability of a version ([19],[24]) of the positive mass theorem to this extension; see [21] for a thorough discussion. We call the pair (γ, H) Bartnik data.

We can now state our results. The definition of m_{out} , like that of the original Bartnik mass, does not directly lend itself to straightforward evaluation. At present its exact value is known only on spherically symmetric Bartnik data ([17]) and on apparent horizons ($H = 0$) satisfying a natural nondegeneracy condition ([18]). Our main theorem gives a first-order estimate for Bartnik data close to $(\iota^* \delta, -2)$.

Theorem 1.6. *There exist $\epsilon, C > 0$ such that the Bartnik-Bray outer mass of (S^2, γ, H) (as defined by equation 1.3) satisfies the estimate*

$$(1.7) \quad \left| m_{out}[\gamma, H] - \int_{S^2} (6 + 2H - \gamma^\mu_\mu) \right| \leq C \|\gamma - \iota^* \delta\|_{C^4}^2 + C \|H + 2\|_{C^3}^2$$

whenever $\|\gamma - \iota^* \delta\|_{C^4} + \|H + 2\|_{C^3} < \epsilon$. Here $\iota^* \delta$ is the standard metric on the unit sphere S^2 , which defines the trace of γ , the measure for the integral, and the norms on the right-hand side.

As an application we get the following asymptotic estimate for the outer mass of small metric spheres.

Corollary 1.8. *Let p be a point in a smooth Riemannian manifold (M, g) . Write R for the scalar curvature of g and B_r for the closed geodesic ball of radius r and center p . Then*

$$(1.9) \quad \lim_{r \rightarrow 0} \frac{m_{out}[B_r, g]}{r^3} = \frac{R(p)}{12},$$

If moreover (M, g) is flat at p , then

$$(1.10) \quad \lim_{r \rightarrow 0} \frac{m_{out}[B_r, g]}{r^5} = \frac{\Delta R(p)}{120},$$

where Δ is the Laplacian defined by g .

Of physical interest is the case when (M, g) arises as time-symmetric initial data for the Einstein equations, where time symmetry means simply that M is totally geodesic in the corresponding spacetime solution and implies that the scalar curvature of M is 16π times the energy density contributed by all fields other than gravity itself. The first limit of the corollary then states that the mass-to-volume ratio of concentric spheres shrinking to a point tends to the energy density at their center. This result can be compared with corresponding ones (such as in [16], [10], [28], [15], and [11]) for other quasilocal masses and is physically natural from the point of view of the equivalence principle, by which nongravitational sources of mass must dominate gravitational ones on small neighborhoods of any given point in the slice.

The corollary is proved (in Section 3) by expanding H and γ^μ_μ in r and applying the theorem. Unfortunately the estimate for the error in the theorem does not preclude the possibility that it might have the same size in vacuum (vanishing scalar curvature) as the leading terms in the

expansion of $H + 2$ and $\gamma^\mu_\mu - 2$, and, without a more refined estimate, we can rule this out only by assuming the metric to be flat at g , as done for the second limit in the corollary. In particular we do not identify the leading (purely gravitational) contribution to the outer mass in vacuum, though this information is available for other quasilocal masses (as in the references just cited).

The theorem itself is proved by identifying a lower bound and an upper bound for the outer mass that agree to first order in the data's deviation from $(\iota^*\delta, -2)$. It is a well-known consequence of [17] that the Hawking mass provides a lower bound. For the upper bound, encouraged by Bartnik's static extension conjecture ([6],[7],[8]), we use the mass of a static vacuum extension.

We recall that a static spacetime (Chapter XIV of [12]) is (by one of several slightly differing definitions) a Lorentzian manifold possessing a complete, timelike, hypersurface-orthogonal Killing field and that a Riemannian metric g on a manifold M is called static ([12], Section 2 of [14]) if (M, g) is a slice orthogonal to such a Killing field in some static spacetime satisfying the Einstein equations, possibly with matter. Equivalently, there is a positive function Φ on M , representing the absolute value of the Lorentzian length of the Killing field, such that the metric $-\Phi^2 dt^2 + g$ on $\mathbb{R} \times M$ is, in the vacuum case, Ricci flat. The evolution and constraint equations in turn imply the intrinsic characterization that (M, g) is static vacuum if and only if there is a function Φ so that together g and Φ solve the static vacuum equations

$$(1.11) \quad \begin{aligned} D_g^2 \Phi - \Phi \operatorname{Ric}[g] &= 0 \\ \Delta_g \Phi &= 0, \end{aligned}$$

where $D_g^2 \Phi$, $\Delta_g \Phi$, and $\operatorname{Ric}[g]$ are respectively the g Hessian of Φ , the g Laplacian of Φ , and the Ricci curvature of g . In this context the lapse function Φ is called a static potential (because for weak fields it approximates the Newtonian gravitational potential) for (M, g) .

We will call the boundary value problem given by the system 1.11 with prescribed Bartnik boundary data the static extension problem. The existence of a solution to the static extension problem in our small-data regime is not a new result. The problem was solved first by Miao in [20], with a symmetry assumption on the boundary data, and later in generality by Anderson (whose proof can be extended to higher dimensions) in [2], applying the framework developed in [1] and [3]. Anderson's existence proof simultaneously establishes uniqueness (up to diffeomorphism) within the class of static metrics (g, Φ) close to $(\delta, 1)$.

In the next section we present a third construction of the extension, which is sufficiently explicit to permit the calculation of the extension's mass to first order in its boundary data's deviation from a standard sphere in Euclidean space. In the final section we apply this estimate in the proof of the main theorem and we end with the proof of the corollary.

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2. THE STATIC VACUUM EXTENSION AND ITS MASS

Let M denote the complement in \mathbb{R}^3 of the open unit ball centered at the origin, equipped with its natural structure as a manifold with boundary, and let $\iota : S^2 \rightarrow M$ be the inclusion map for the boundary $\partial M = S^2$, the origin-centered unit sphere. Writing $Met_{loc}^2(\cdot)$ for the space of C_{loc}^2 Riemannian metrics on a manifold with (possibly empty) boundary, we define the operators $\mathcal{S} : Met_{loc}^2(M) \times C_{loc}^2(M) \rightarrow C_{loc}^0(T^*M^{\odot 2}) \times C_{loc}^0(M)$ and $\mathcal{B} : Met_{loc}^2(M) \rightarrow Met_{loc}^2(\partial M) \times C_{loc}^1(\partial M)$ by

$$(2.1) \quad \mathcal{S} \begin{bmatrix} g \\ \Phi \end{bmatrix} = \begin{bmatrix} D_g^2 \Phi - \Phi \operatorname{Ric}[g] \\ \Delta_g \Phi \end{bmatrix} \quad \text{and} \quad \mathcal{B}[g] = \begin{bmatrix} \iota^* g \\ \mathcal{H}_\iota[g] \end{bmatrix},$$

where $\text{Ric}[g]$, D_g , and Δ_g respectively denote the Ricci curvature, Levi-Civita connection, and Laplacian of g , and where $\mathcal{H}_\iota[g]$ denotes the scalar mean curvature of ∂M relative to g and the corresponding unit normal directed toward the interior of M . Then g and Φ solve the static vacuum extension problem on M with boundary data $\gamma \in \text{Met}_{loc}^2(\partial M)$ and $H \in C_{loc}^1(\partial M)$ if and only if

$$(2.2) \quad \mathcal{S} \begin{bmatrix} g \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \mathcal{B}[g] = \begin{bmatrix} \gamma \\ H \end{bmatrix}.$$

Of course the Euclidean metric δ on M is static, with potential $\Phi \equiv 1$, so for boundary metric γ close to the round metric ι^*g and boundary mean curvature H close to -2 , we seek an extension of the form $g = \delta + \eta$ with $\Phi = 1 + u$, where η is a small section of $T^*M^{\odot 2}$ decaying rapidly enough for g to be asymptotically flat and u is similarly a small function with appropriate decay.

Given a section F of a tensor bundle over \mathbb{R}^3 or M or ∂M , a nonnegative integer j , $\alpha \in [0, 1)$, and a positive real number β , we will make use of the Hölder norm

$$(2.3) \quad \|F\|_{j,\alpha} = \sup_{\vec{x} \neq \vec{y} \in M} \frac{|D^j F(\vec{x}) - D^j F(\vec{y})|_\delta}{|\vec{x} - \vec{y}|^\alpha} + \sum_{i=0}^j \sup_{\vec{x} \in M} |D_\delta^i F(\vec{x})|_\delta$$

as well as the weighted Hölder norm

$$(2.4) \quad \|F\|_{j,\alpha,\beta} = \sup_{\vec{x} \neq \vec{y} \in M} [1 + \min\{|\vec{x}|, |\vec{y}|\}]^{\alpha+\beta+j} \frac{|D^j F(\vec{x}) - D^j F(\vec{y})|_\delta}{|\vec{x} - \vec{y}|^\alpha} + \sum_{i=0}^j \sup_{\vec{x} \in M} (1 + |\vec{x}|)^{\beta+i} |D_\delta^i F(\vec{x})|_\delta.$$

For each tensor bundle E over M (or ∂M) we also define the Banach spaces $C^{j,\alpha,\beta}(E)$ and $C^{j,\alpha}(E)$ (written simply $C^{j,\alpha,\beta}(M)$ and $C^{j,\alpha}(M)$ as usual when E is the trivial bundle $M \times \mathbb{R}$ (and likewise for $\partial M \times \mathbb{R}$)) of sections of E with finite $\|\cdot\|_{j,\alpha,\beta}$ and $\|\cdot\|_{j,\alpha}$ norms respectively.

Linearizing the above operators about $g = \delta$ and $\Phi = 1$ we define

$$(2.5) \quad \begin{aligned} \dot{\mathcal{S}} : C^{j+2,\alpha,\beta}(T^*M^{\odot 2}) \oplus C^{j+2,\alpha,\beta}(M) &\rightarrow C^{j,\alpha,\beta+2}(T^*M^{\odot 2}) \oplus C^{j,\alpha,\beta+2}(M) \text{ by} \\ \dot{\mathcal{S}} \begin{bmatrix} \eta \\ u \end{bmatrix} &:= \left. \frac{d}{dt} \right|_{t=0} \mathcal{S} \begin{bmatrix} \delta + t\eta \\ 1 + tu \end{bmatrix} = \begin{bmatrix} D_\delta^2 u - \text{Ric}[\eta] \\ \Delta_\delta u \end{bmatrix} \end{aligned}$$

and

$$(2.6) \quad \begin{aligned} \dot{\mathcal{B}} : C^{j+2,\alpha,\beta}(T^*M^{\odot 2}) &\rightarrow C^{j+2,\alpha}(T^*\partial M^{\odot 2}) \oplus C^{j+1,\alpha}(\partial M) \text{ by} \\ \dot{\mathcal{B}}[\eta] &:= \left. \frac{d}{dt} \right|_{t=0} \mathcal{B}[\delta + t\eta] = \begin{bmatrix} \iota^* \eta \\ \dot{\mathcal{H}}_\iota[\eta] \end{bmatrix}, \end{aligned}$$

where $\dot{\text{Ric}}[\eta] = \left. \frac{d}{dt} \right|_{t=0} \text{Ric}[\delta + t\eta]$ and $\dot{\mathcal{H}}_\iota[\eta] = \left. \frac{d}{dt} \right|_{t=0} \mathcal{H}_\iota[\delta + t\eta]$ are the linearization about δ , with respect to ambient metric, of the Ricci curvature of M and the mean curvature of ∂M respectively.

The following lemma, whose second item will ultimately furnish the mass estimate, is our principal tool for solving the extension problem. The heart of the proof is the recognition that, at the linear level, the static condition can be maintained while making arbitrary perturbations to the boundary data through harmonic conformal transformations and diffeomorphisms from M to subsets of \mathbb{R}^3 that distort the boundary.

Lemma 2.7. *Given $\alpha \in (0, 1)$ and $\beta \in (0, 1) \cup (1, 2)$, there exists a bounded linear map*

$$(2.8) \quad \begin{aligned} \mathcal{U} : C^{1,\alpha,\beta+2}(T^*M^{\odot 2}) \oplus C^{1,\alpha,\beta+2}(M) \oplus C^{3,\alpha}(T^*\partial M^{\odot 2}) \oplus C^{2,\alpha}(\partial M) \\ \rightarrow C^{3,\alpha,\beta'}(T^*M^{\odot 2}) \oplus C^{3,\alpha,\beta'}(M) \oplus C^{2,\alpha,\beta'+1}(TM), \end{aligned}$$

where $\beta' = \min\{\beta, 1\}$, such that if

$$(2.9) \quad \begin{bmatrix} \eta_{ab} \\ u \\ \chi^a \end{bmatrix} = \mathcal{U} \begin{bmatrix} S_{ab} \\ \sigma \\ \omega_{\mu\nu} \\ \kappa \end{bmatrix},$$

then (i)

$$(2.10) \quad \dot{\mathcal{S}} \begin{bmatrix} \eta_{ab} \\ u \end{bmatrix} = \begin{bmatrix} S_{ab} + \chi_{a;b} + \chi_{b;a} \\ \sigma \end{bmatrix} \quad \text{and} \quad \dot{\mathcal{B}}[\eta_{ab}] = \begin{bmatrix} \omega_{\mu\nu} \\ \kappa \end{bmatrix},$$

where $\dot{}$ and $;$ indicates covariant differentiation relative to δ , and

(ii) in the special case that $S_{ab} = 0$ and $\sigma = 0$, we have $\eta_{ab} = -2u\delta_{ab}$ outside a compact set and moreover

$$(2.11) \quad \int_{\partial M} u = \frac{1}{4} \int_{\partial M} \omega_{\mu}{}^{\mu} - \frac{1}{2} \int_{\partial M} \kappa,$$

where ω is contracted via $\iota^*\delta$ and the integration measure is given by the standard area form on $S^2 = \partial M$ (for a total area of 4π), induced by $\iota^*\delta$.

Proof. (i) Let $S \in C^{1,\alpha,\beta+2}(T^*M^{\odot 2})$, $\sigma \in C^{1,\alpha,\beta+2}(M)$, $\omega \in C^{3,\alpha}(T^*\partial M^{\odot 2})$, and $\kappa \in C^{2,\alpha}(\partial M)$. A necessary condition for a symmetric 2-tensor to be $\text{Ric}[\eta]$ for some η is that it satisfy the linearization at δ of the twice contracted Bianchi identity, namely that its divergence equal half the differential of its trace. It is this requirement which necessitates the introduction of χ .

In more detail, to solve the linearized problem 2.10 without χ we would need η to satisfy $\text{Ric}[\eta]_{ab} = u_{;ab} - S_{ab}$ with u solving the Poisson equation $\Delta_{\delta}u = \sigma$. The above identity would then require $\frac{1}{2}\sigma_{;a} + \frac{1}{2}S^c{}_{c;a} - S_{ab}{}^{;b} = 0$, which of course does not hold for all data. We can achieve the necessary condition though at the cost of adding to S_{ab} the Lie derivative $\chi_{a;b} + \chi_{b;a}$ of δ along a vector field χ^a selected for this purpose. (Such adjustment by infinitesimal diffeomorphism is the same, well-known device used, in the Lorentzian setting, to achieve the Lorenz gauge condition in linearized gravity, though there it is applied to the perturbation rather than to the source.) Specifically we choose χ to solve

$$(2.12) \quad \chi_{a;b}{}^{;b} = \frac{1}{2}\sigma_{;a} + \frac{1}{2}S^c{}_{c;a} - S_{ab}{}^{;b}.$$

Rather than directly imposing boundary data we find it convenient to pick the solution

$$(2.13) \quad \chi_a = G * \left(\frac{1}{2}(E\sigma)_{;a} + \frac{1}{2}(ES)^c{}_{c;a} - (ES)_{ab}{}^{;b} \right),$$

where $*$ denotes Cartesian-componentwise convolution, G is the Newtonian point-source potential $G(\vec{x}) = -\frac{1}{4\pi|\vec{x}|}$, and $E : C^{1,\alpha}(M) \rightarrow C^{1,\alpha}(\mathbb{R}^3)$ is a fixed bounded extension operator (satisfying $(Eu)|_M = u$) that acts on tensor fields Cartesian-componentwise. That $\|\chi\|_{2,\alpha,\beta'+1}$ is bounded by the data as required can be confirmed by direct estimation of convolution against G (as in the classical proof of the Schauder estimates but keeping track of the decay) or by obtaining the $C^{0,0,\beta}$ estimate in this fashion (shown in Appendix A) and then applying the standard interior Schauder estimates along with a scaling argument, as for example in the appendix of [25].

Now, setting

$$(2.14) \quad \tilde{S}_{ab} = ES_{ab} + \chi_{a;b} + \chi_{b;a},$$

we see we have arranged

$$(2.15) \quad \left(\tilde{S}_{ab} - \frac{1}{2}\tilde{S}^c{}_{c;a} \right)^{;b} = \frac{1}{2}(E\sigma)_{;a}.$$

Defining also

$$(2.16) \quad \begin{aligned} w &= G * E\sigma \text{ and} \\ \tilde{\eta}_{ab} &= -2G * \left(w_{;ab} - \tilde{S}_{ab} \right), \end{aligned}$$

now w solves the Poisson equation $\Delta_\delta w = E\sigma$ and $w_{;ab} - \tilde{S}_{ab}$ satisfies the linearized twice contracted Bianchi identity

$$(2.17) \quad \left(w_{;ab} - \tilde{S}_{ab} - \frac{1}{2} w_{;c}{}^{;c} \delta_{ab} + \frac{1}{2} \tilde{S}_c{}^c \delta_{ab} \right)^{;b} = 0,$$

so $\tilde{\eta}$ does too:

$$(2.18) \quad \left(\tilde{\eta}_{ab} - \frac{1}{2} \tilde{\eta}_c{}^c \delta_{ab} \right)^{;b} = 0.$$

Consequently

$$(2.19) \quad \text{Ric}[\tilde{\eta}] = \frac{1}{2} \left(\tilde{\eta}^c{}_{a;cb} + \tilde{\eta}^c{}_{b;ca} - \tilde{\eta}_{ab;c}{}^{;c} - \tilde{\eta}^c{}_{c;ab} \right) = -\frac{1}{2} \Delta_\delta \tilde{\eta} = D_\delta^2 w - \tilde{S}.$$

Thus far we have secured

$$(2.20) \quad \dot{S} \begin{bmatrix} \tilde{\eta} \\ w \end{bmatrix} = \begin{bmatrix} \tilde{S} \\ \sigma \end{bmatrix},$$

with $\|w\|_{3,\alpha,\beta'}$ and $\|\tilde{\eta}\|_{3,\alpha,\beta'}$ controlled by the data, but it remains to enforce the boundary conditions.

To this end we will further alter the metric by infinitesimal diffeomorphism of the form $\xi_{a;b} + \xi_{b;a}$ with ξ^a a compactly supported vector field, which of course does not affect the linearized Ricci tensor, so preserves equation 2.20, but can be exploited to adjust the boundary data. Fields tangential to ∂M represent pure gauge changes, which are nevertheless useful for reducing to the case of boundary metric conformally round to first order (infinitesimal uniformization), while normal fields genuinely deform the boundary geometry.

In fact we restrict attention to vector fields of the form

$$(2.21) \quad \xi = \psi \xi^\perp \partial_r + \psi \xi^\top,$$

where ξ^\perp is a (scalar-valued) function on ∂M , ξ^\top is a radially constant (δ -parallel along rays through the origin) vector field tangential to ∂M , and ψ is a bump function identically 1 on $\{|x| \leq 2\}$ and identically 0 on $\{|x| \geq 3\}$. Thus we consider variations of ∂M arising as reparametrizations generated by ξ^\top and as graphs generated by ξ^\perp . To wit, writing L for the Lie derivative, we have

$$(2.22) \quad \dot{B}[\xi_{a;b} + \xi_{b;a}] = \begin{bmatrix} L_{\xi^\top} \iota^* \delta + 2\xi^\perp \iota^* \delta \\ (\Delta_{\iota^* \delta} + 2) \xi^\perp \end{bmatrix},$$

as can be verified either by direct calculation of the variation of the induced metric and mean curvature under variations of the ambient metric (included in Appendix B for reference) or by using the possibly more familiar formulas for the variation of these quantities under variations of the immersion, after applying the tautologous identities $\iota^* \phi_t^* \delta = (\phi_t \circ \iota)^* \delta$ and $\mathcal{H}_\iota[\phi_t^* \delta] = \mathcal{H}_{\phi_t \circ \iota}[\delta]$ for the diffeomorphisms $\phi_t : M \rightarrow \mathbb{R}^3$ generated by ξ .

Of course variations by vector fields alone will not span the tangent space of boundary data at δ , and so we will additionally avail ourselves of linearized conformal transformations generated by harmonic (relative to the Euclidean metric δ) functions vanishing at infinity. For v satisfying $\Delta_\delta v = 0$ we get

$$(2.23) \quad \text{Ric}[v\delta] = -\frac{1}{2} D_\delta^2 v,$$

so we may simultaneously replace $\tilde{\eta}$ and w in 2.20 by $\tilde{\eta} + v\delta$ and $w - \frac{1}{2}v$ respectively, again without altering the right-hand side. As for the boundary data, the change of the induced metric is obvious and the change of mean curvature is easy to calculate (as shown in Appendix B):

$$(2.24) \quad \dot{\mathcal{B}}[v\delta] = \begin{bmatrix} v\iota^*\delta \\ v - v_{,r} \end{bmatrix}.$$

We will soon see that these two types of modifications will suffice to prescribe the boundary data, and accordingly we seek ξ^a and v as just described so that

$$(2.25) \quad \dot{\mathcal{B}}[\tilde{\eta}_{ab} + \xi_{a;b} + \xi_{b;a} + v\delta_{ab}] = \begin{bmatrix} \omega_{\mu\nu} \\ \kappa \end{bmatrix}.$$

We write $\dot{\mathcal{B}}_1 : C^{j,\alpha,\beta}(T^*M^{\odot 2}) \rightarrow C^{j,\alpha}(T^*\partial M^{\odot 2})$ and $\dot{\mathcal{B}}_2 : C^{j+1,\alpha,\beta}(T^*M^{\odot 2}) \rightarrow C^{j,\alpha}(\partial M)$ for the first (metric) and second (mean curvature) components respectively of $\dot{\mathcal{B}}$.

Since (see for example Lectures 1 and 3 of [27]) every symmetric 2-tensor on S^2 may be written as the sum of a Lie derivative of $\iota^*\delta$ and a function times $\iota^*\delta$, there exist $h \in C^{3,\alpha}(\partial M)$ and $W \in C^{4,\alpha}(T\partial M)$ such that

$$(2.26) \quad \omega - \dot{\mathcal{B}}_1[\tilde{\eta}] = h\iota^*\delta + L_W\iota^*\delta.$$

We also define $k \in C^{2,\alpha}(\partial M)$ by

$$(2.27) \quad \kappa - \dot{\mathcal{B}}_2[\tilde{\eta}] = k.$$

Setting $\xi^\top = W$ and referring to equations 2.22 and 2.24, equation 2.25 becomes the system

$$(2.28) \quad \begin{aligned} v + 2\xi^\perp &= h \\ v - v_{,r} + (\Delta_{\iota^*\delta} + 2)\xi^\perp &= k. \end{aligned}$$

For each nonnegative integer ℓ and for each integer $m \in [-\ell, \ell]$ we let $Y_{\ell,m} : S^2 \rightarrow \mathbb{C}$ be a spherical harmonic satisfying $\Delta_{\iota^*\delta}Y_{\ell,m}(\theta, \phi) = -\ell(\ell+1)Y_{\ell,m}(\theta, \phi)$ and $-i\partial_\theta Y_{\ell,m}(\theta, \phi) = mY_{\ell,m}(\theta, \phi)$ (where θ is the azimuth angle) and chosen so that $\bigcup_{\ell=0}^\infty \{Y_{\ell,m}\}_{m=-\ell}^\ell$ is an orthonormal basis for $L^2(S^2, \iota^*\delta)$.

Introducing the coefficients defined by the expansions

$$(2.29) \quad \begin{aligned} h(\theta, \phi) &= \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell h_{\ell,m} Y_{\ell,m}(\theta, \phi), \\ k(\theta, \phi) &= \sum_{\ell,m} k_{\ell,m} Y_{\ell,m}(\theta, \phi), \\ v(r, \theta, \phi) &= \sum_{\ell,m} v_{\ell,m} r^{-\ell-1} Y_{\ell,m}(\theta, \phi), \text{ and} \\ \xi^\perp(r, \theta, \phi) &= \sum_{\ell,m} \xi_{\ell,m} Y_{\ell,m}(\theta, \phi), \end{aligned}$$

we find that our system 2.28 is satisfied if and only if

$$(2.30) \quad \begin{bmatrix} 1 & 2 \\ \ell+2 & -(\ell-1)(\ell+2) \end{bmatrix} \begin{bmatrix} v_{\ell,m} \\ \xi_{\ell,m} \end{bmatrix} = \begin{bmatrix} h_{\ell,m} \\ k_{\ell,m} \end{bmatrix}$$

for each nonnegative integer ℓ and for each integer $m \in [-\ell, \ell]$. Evidently for each such ℓ the matrix in equation 2.30 is invertible, so we get the unique solution

$$(2.31) \quad \begin{bmatrix} v \\ \xi \end{bmatrix}_{\ell, m} = \begin{bmatrix} \frac{\ell-1}{\ell+1} & \frac{2}{(\ell+1)(\ell+2)} \\ \frac{1}{\ell+1} & \frac{-1}{(\ell+1)(\ell+2)} \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}_{\ell, m}.$$

Since $h \in C^{3,\alpha}(\partial M)$ and $k \in C^{2,\alpha}(\partial M)$, the coefficients $h_{\ell, m}$ and $k_{\ell, m}$ are uniformly bounded (and in fact decay), so the coefficients $v_{\ell, m}$ and $\xi_{\ell, m}$ given by 2.31 certainly define at least a distributional solution to the system 2.25. The regularity and estimates required of v and ξ can then be deduced from the representations

$$(2.32) \quad \begin{aligned} v(r, \theta, \phi) &= P[h](r, \theta, \phi) - 2 \int_r^\infty s^{-1} P[h](s, \theta, \phi) ds + 2 \int_r^\infty \int_s^\infty t^{-2} P[k](t, \theta, \phi) dt ds \text{ and} \\ \xi^\perp(r, \theta, \phi) &= \int_r^\infty s^{-1} P[h](s, \theta, \phi) ds - \int_r^\infty \int_s^\infty t^{-2} P[k](t, \theta, \phi) dt ds, \end{aligned}$$

where $P[f]$ denotes the harmonic extension of a given $f \in C^0(\partial M)$ to a (δ) -harmonic function on M vanishing at ∞ , itself representable by

$$(2.33) \quad P[f](\vec{x}) = \int_{S^2} P(\vec{x}, \vec{y}) f(\vec{y}),$$

with exterior Poisson kernel

$$(2.34) \quad P(\vec{x}, \vec{y}) = \frac{|\vec{x}|^2 - 1}{4\pi |\vec{x} - \vec{y}|^3}.$$

The proof of (i) is now complete, with the solution operator defined by

$$(2.35) \quad \mathcal{U} \begin{bmatrix} S_{ab} \\ \sigma \\ \omega_{\mu\nu} \\ \kappa \end{bmatrix} = \begin{bmatrix} \tilde{\eta}_{ab} + \xi_{a;b} + \xi_{b;a} + v\delta_{ab} \\ w - \frac{1}{2}v \\ \chi^a \end{bmatrix}.$$

(ii) We first observe that when S_{ab} and σ vanish identically, so do χ^a , w , and $\tilde{\eta}$, recalling equations 2.13 and 2.16, so that equations 2.26 and 2.27 read simply $\omega_{\mu\nu} = h(\iota^*\delta)_{\mu\nu} + W_{\mu;\nu} + W_{\nu;\mu}$ and $\kappa = k$, with $:$ indicating covariant differentiation according to $\iota^*\delta$. Thus in particular $\omega_\mu{}^\mu = 2h + 2W_\mu{}^\mu$, so $\int_{S^2} \omega_\mu{}^\mu = 2 \int_{S^2} h$ and of course $\int_{S^2} \kappa = \int_{S^2} k$. Referring to the linearized potential $u = 0 - \frac{1}{2}v$ appearing in 2.35 and to the top row of 2.31 at $\ell = m = 0$, we finish with

$$(2.36) \quad \int_{S^2} u = -\frac{1}{2} \int_{S^2} v = \frac{1}{2} \int_{S^2} h - \frac{1}{2} \int_{S^2} k = \frac{1}{4} \int_{S^2} \omega_\mu{}^\mu - \frac{1}{2} \int_{S^2} \kappa.$$

□

Taking $S = 0$, $\sigma = 0$, $\omega = \gamma - \iota^*\delta$, and $\kappa = H + 2$ in the lemma, from \mathcal{U} we obtain η_{ab} and u (and $\chi = 0$) solving our extension problem to first order:

$$(2.37) \quad \dot{S} \begin{bmatrix} \eta \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \dot{\mathcal{B}}[\eta] = \begin{bmatrix} \gamma - \iota^*\delta \\ H + 2 \end{bmatrix}.$$

The proof of the next proposition provides the higher-order corrections by using the contraction mapping lemma. Note that this first-order application of 2.7 requires no χ^a to adjust the source terms, but subsequent applications, aimed to cancel the nonlinear terms, a priori may, spoiling the static condition. To the contrary, an argument along the lines of Miao's proof of his Reduction

Lemma in [20] shows that for sufficiently small data χ^a vanishes identically, leaving an exact solution.

The existence and uniqueness assertions of the proposition were established by Anderson in [2], and the analyticity, relative to harmonic local coordinates, of static vacuum metrics was proved by Müller zum Hagen in [22]. The novelty of the proposition is its estimate of the extension's ADM mass, which is made by referring to item (ii) of the lemma to compute the mass of the linearized solution and by using the estimate for the nonlinear corrections obtained in the course of constructing the exact solution.

Proposition 2.38. *Let $\alpha \in (0, 1)$. There exist $C, \epsilon > 0$ such that whenever $\|\gamma_{\mu\nu} - (\iota^*\delta)_{\mu\nu}\|_{3,\alpha} + \|H + 2\|_{2,\alpha} < \epsilon$, (i) there is a $C^{3,\alpha}$ asymptotically flat Riemannian metric g on M and there is a $C^{3,\alpha}$ function Φ on M solving the static vacuum extension problem*

$$(2.39) \quad \mathcal{S} \begin{bmatrix} g \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathcal{B}[g] = \begin{bmatrix} \gamma_{\mu\nu} \\ H \end{bmatrix},$$

with

$$(2.40) \quad \|g - \delta\|_{3,\alpha,1} + \|\Phi - 1\|_{3,\alpha,1} \leq C \|\gamma - \iota^*\delta\|_{3,\alpha} + C \|H + 2\|_{2,\alpha};$$

(ii) there is a neighborhood of $(\delta, 1)$ in $C^{3,\alpha,\beta}(T^*M^{\odot 2}) \times C^{3,\alpha,\beta}(M)$ within which any two solutions of 2.39 are diffeomorphic; (iii) there are global harmonic coordinates for the interior of (M, g) with respect to which g and Φ are analytic; (iv) (g, Φ) admits no closed minimal surfaces and ∂M is outer-minimizing; and (v) the ADM mass $m_{ADM}[g]$ of the extension g satisfies the estimate

$$(2.41) \quad m_{ADM}[g] = \frac{1}{16\pi} \int_{\partial M} (6 + 2H - \gamma_\mu^\mu) + O(\|\gamma - \iota^*\delta\|_{3,\alpha}^2) + O(\|H + 2\|_{2,\alpha}^2).$$

Proof. (i) We seek a solution of the form $g_{ab} = \delta_{ab} + \eta_{ab} + \theta_{ab}$ and $\Phi = 1 + u + v$, with η and u obtained from Lemma 2.7 as in 2.37 and θ_{ab} and v to be determined. Fixing some $\beta \in (\frac{1}{2}, 1)$, we get that the $C^{3,\alpha,\beta}$ norms of η and u are bounded by $\|U\|$ times the norm of the data; in fact, referring to item (ii) of the lemma, since u is harmonic, the $C^{3,\alpha,1}$ norms of η and u are likewise bounded.

Now we need to eliminate the nonlinear errors defined by

$$(2.42) \quad \mathcal{Q}_S \begin{bmatrix} \eta_{ab} \\ u \end{bmatrix} := \mathcal{S} \begin{bmatrix} \delta + \eta \\ 1 + u \end{bmatrix} - \dot{\mathcal{S}} \begin{bmatrix} \eta \\ u \end{bmatrix} \quad \text{and} \quad \mathcal{Q}_B[\eta_{ab}] := \mathcal{B}[\delta + \eta] - \mathcal{B}[\delta] - \dot{\mathcal{B}}[\eta].$$

that our first-order solutions introduce, and accordingly we will make a standard application of the contraction mapping lemma to secure a fixed point to the map

$$(2.43) \quad \begin{bmatrix} \theta_{ab} \\ v \end{bmatrix} \mapsto -\pi \mathcal{U} \begin{bmatrix} \mathcal{Q}_S \begin{bmatrix} \eta_{ab} + \theta_{ab} \\ u + v \end{bmatrix} \\ \mathcal{Q}_B[\eta_{ab} + \theta_{ab}] \end{bmatrix},$$

where π projects onto the first two factors of the target of \mathcal{U} (forgetting the vector field χ used to modify the source S).

First we observe that the induced metric on ∂M is independent of the potential Φ and linear in the ambient metric g , while, for g near δ , the mean curvature (being the divergence along ∂M of the g -normalized vector field g -dual to the 1-form dr) depends smoothly, as a function of some local coordinates on ∂M , on the Cartesian components g_{ij} of g and on its first $g_{ij,k}$ and second $g_{ij,kl}$ partial derivatives. Consequently, taking ϵ sufficiently small, we have

$$(2.44) \quad \|\mathcal{Q}_B[\eta_{ab} + \theta_{ab}]\|_{2,\alpha} = \left\| \int_0^1 \int_0^s \frac{d^2}{ds^2} \mathcal{B}[\delta + s(\eta_{ab} + \theta_{ab})] ds dt \right\|_{2,\alpha} \leq C_1 \|\eta + \theta\|_{3,\alpha}^2 \quad \text{and}$$

(2.45)

$$\begin{aligned} \left\| \mathcal{Q}_{\mathcal{B}}[\eta_{ab} + \theta_{ab}] - \mathcal{Q}_{\mathcal{B}}[\eta_{ab} + \tilde{\theta}_{ab}] \right\|_{2,\alpha} &= \left\| \int_0^1 \int_0^s \int_0^1 \frac{\partial^3}{\partial s^2 \partial \tau} \mathcal{B}[\delta + s(\eta + \tilde{\theta}) + s\tau(\theta - \tilde{\theta})] d\tau ds dt \right\|_{2,\alpha} \\ &\leq C_1 \|\eta + \theta\|_{3,\alpha} \|\theta - \tilde{\theta}\|_{3,\alpha} \end{aligned}$$

for some $C_1 > 0$.

We also observe, working in Cartesian coordinates, that the components of \mathcal{S} are polynomial in the components g_{ij} of the ambient metric, the components g^{ij} of its inverse, its first $g_{ij,k}$ and second $g_{ij,kl}$ partials, the potential Φ , and its first $\Phi_{,i}$ and second $\Phi_{,ij}$ partials, and so we similarly have for some $C_2 > 0$ and ϵ sufficiently small

$$(2.46) \quad \left\| \mathcal{Q}_{\mathcal{S}} \begin{bmatrix} \eta_{ab} + \theta_{ab} \\ u + v \end{bmatrix} \right\|_{1,\alpha,2\beta+2} \leq C_2 (\|\eta + \theta\|_{3,\alpha,\beta} + \|u + v\|_{3,\alpha,\beta})^2$$

and

$$(2.47) \quad \left\| \mathcal{Q}_{\mathcal{S}} \begin{bmatrix} \eta_{ab} + \theta_{ab} \\ u + v \end{bmatrix} - \mathcal{Q}_{\mathcal{S}} \begin{bmatrix} \eta_{ab} + \tilde{\theta}_{ab} \\ u + \tilde{v} \end{bmatrix} \right\|_{1,\alpha,\beta+2} \leq C_2 \left(\|\eta + \theta\|_{3,\alpha,\beta} + \|u + v\|_{3,\alpha,\beta} \right) \left(\|\theta - \tilde{\theta}\|_{3,\alpha,\beta} + \|v - \tilde{v}\|_{3,\alpha,\beta} \right).$$

Thus, choosing $C_3 > 0$ sufficiently large and $\epsilon > 0$ sufficiently small—both in terms of C_1 , C_2 , and the operator norm of \mathcal{U} —we see that the map given by 2.43 is a contraction from

$$(2.48) \quad \left\{ \|\theta_{ab}\|_{3,\alpha,\beta} + \|v\|_{3,\alpha,\beta} \leq C_3 \left(\|\gamma - \iota^* \delta\|_{3,\alpha} + \|H + 2\|_{2,\alpha} \right)^2 \right\}$$

to itself. By taking (θ_{ab}, v) to be this map's unique fixed point and

$$(2.49) \quad \chi^a = -\pi_3 \mathcal{U} \begin{bmatrix} \mathcal{Q}_{\mathcal{S}} \begin{bmatrix} \eta_{ab} + \theta_{ab} \\ u + v \end{bmatrix} \\ \mathcal{Q}_{\mathcal{B}}[\eta_{ab} + \theta_{ab}] \end{bmatrix},$$

where π_3 is projection onto the third factor of the target of \mathcal{U} , we obtain (g, Φ) solving

$$(2.50) \quad \mathcal{S} \begin{bmatrix} g_{ab} \\ \Phi \end{bmatrix} = \begin{bmatrix} \chi_{a;b} + \chi_{b;a} \\ 0 \end{bmatrix} \text{ and } \mathcal{B}[g_{ab}] = \begin{bmatrix} \gamma_{ab} \\ H \end{bmatrix}.$$

Note that since θ is a component of a fixed point of 2.43, on account of the bound 2.46 and Lemma 2.7 with exponent of decay exceeding 1 we get for some $C_4 > 0$

$$(2.51) \quad \|\theta_{ab}\|_{3,\alpha,1} \leq C_5 \left(\|\gamma - \iota^* \delta\|_{3,\alpha} + \|H + 2\|_{2,\alpha} \right)^2.$$

Now, writing R_{ab} and R for the Ricci and scalar curvature of g , we have achieved

$$(2.52) \quad \begin{aligned} \Phi_{|ab} - \Phi R_{ab} &= \chi_{a;b} + \chi_{b;a} \\ \Phi_{|c} &= 0, \end{aligned}$$

where $|$ indicates covariant differentiation relative to g (and $;$ continues to indicate covariant differentiation relative to δ). Mimicking [20], we derive from 2.52 a linear elliptic system for χ , with coefficients depending on g and Φ . For brevity we set

$$(2.53) \quad \tau_{ab} = \chi_{a;b} + \chi_{b;a} \in C^{1,\alpha,\beta+2}(T^*M^{\odot 2}),$$

the right-hand side of the first equation in 2.52. Then, taking the g trace of the first equation, using the second, rearranging, and taking a derivative, we get

$$(2.54) \quad R_{|a} = -\Phi^{-1} g^{cd} \tau_{cd|a} + \Phi_{|a} \Phi^{-2} g^{cd} \tau_{cd},$$

where g^{cd} denotes g^{-1} (rather than contraction of g with δ^{-1}), while the g divergence of the first gives

$$(2.55) \quad -\Phi R_{ab}{}^{;b} = \tau_{ab}{}^{;b}.$$

Combining these last two equations via the twice contracted Bianchi identity we arrive at

$$(2.56) \quad \tau_{ab}{}^{;b} - \frac{1}{2}g^{cd}\tau_{cd|a} + \frac{1}{2}(\ln \Phi)_{|a}g^{cd}\tau_{cd} = 0,$$

which we rewrite as

$$(2.57) \quad \chi_{a;b}{}^{;b} = B^b\tau_{ab} + C^{cd}{}_a\tau_{cd} + D^{cd}\tau_{cd;a}$$

for tensor fields on M satisfying $\|B\|_{2,\alpha,\beta+1} + \|C\|_{2,\alpha,\beta+1} + \|D\|_{3,\alpha,\beta} \leq C_5\epsilon$, for some $C_5 > 0$ independent of ϵ , but the determination of χ by equations 2.49 and 2.13 then implies the bound

$$(2.58) \quad \|\chi\|_{2,\alpha,\beta+1} \leq C_6\epsilon \|\chi\|_{2,\alpha,\beta+1}$$

for some $C_6 > 0$ independent of ϵ , which for ϵ small enough forces $\chi \equiv 0$. Thus g is exactly static with nowhere vanishing potential Φ . The estimates of g_{ab} and Φ follow from the above bounds for η_{ab} , u , θ_{ab} , and v , completing the proof of (i).

(ii) The uniqueness can be established by a contradiction argument, appealing again to the contractiveness of the nonlinear terms established above and studying the linearized problem with trivial data, but, now that we have in hand a solution with estimates adequate for our application, we refer the reader to [2] for a proof of uniqueness, which will not be needed in this article.

(iii) For the analyticity see [22] or Proposition 2.8 in [14]. A proof of the existence of a harmonic coordinate system near infinity for an arbitrary asymptotically flat metric can be found in [5] for instance. In the present, nearly-Euclidean setting, if $\{x^i\}$ are Cartesian coordinates on \mathbb{R}^3 (restricted to M), the estimate for $g - \delta$ ensures that $\Delta_g x^i$ can be made arbitrarily small in $C^{0,\alpha,2+\beta}$ by taking ϵ small, so, picking a bounded right inverse $\tilde{G} : C^{0,\alpha,2+\beta} \rightarrow C^{2,\alpha,\beta}$ for Δ_g , we get g -harmonic coordinates $\{x^i - \tilde{G}\Delta_g x^i\}$ on the interior of M .

(iv) The absence of closed minimal surfaces likewise follows from the smallness of $g - \delta$ and from the maximum principle. Indeed the function r^2 on M of course has δ -Hessian $(r^2)_{;ab} = 2\delta_{ab}$ and g -Hessian $(r^2)_{|ab} = (r^2)_{;ab} - r(g_{ar;b} + g_{br;a} - g_{ab;r})$. For ϵ sufficiently small then r^2 is everywhere strictly convex on (M, g) , so its restriction to any minimal surface in (M, g) is subharmonic and as such can attain a maximum value only on its boundary. Thus M is devoid of closed minimal surfaces, and similarly if Σ is any least-area surface in M homologous to ∂M , then off ∂M it is a properly embedded minimal surface, so in fact coincides with ∂M , which is thereby outer-minimizing.

(v) For the mass estimate we first observe that

$$(2.59) \quad m_{ADM}[g] = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{|\vec{x}|=r} (\eta_{ra}{}^{;a} - \eta^c{}_{cr}) + O\left(\|\gamma - \iota^*\delta\|_{3,\alpha}^2 + \|H + 2\|_{2,\alpha}^2\right)$$

in light of the bound 2.51. By item (ii) of Lemma 2.7 the linearized metric η is conformally flat at infinity with harmonic conformal factor $-2u$, so that, using equation 2.11, for large r

$$(2.60) \quad \begin{aligned} \int_{|\vec{x}|=r} (\eta_{ra}{}^{;a} - \eta^c{}_{cr}) &= 4 \int_{|\vec{x}|=r} u_{,r} = 4 \int_{|\vec{x}|=1} u_{,r} = -4 \int_{\partial M} u \\ &= \int_{\partial M} [2(H + 2) - (\gamma - \iota^*\delta)_\mu{}^\mu]. \end{aligned}$$

□

3. PROOF OF THE MAIN THEOREM AND ITS COROLLARY

Proof of Theorem 1.6. To prove the theorem, take ϵ as in Proposition 2.38 (say with $\alpha = \beta = 3/4$), $\gamma \in C^4(T^*\partial M^{\odot 2})$, $H \in C^3(\partial M)$ satisfying $\|\gamma - \iota^*\delta\|_{C^4} + \|H + 2\|_{C^3} < \epsilon$, and let g be the corresponding extension guaranteed by item (i) of the proposition. Then $g \in \mathcal{PM}[\gamma, H]$ as defined (1.2) in the introduction and by item (iv) of the proposition ∂M is outer-minimizing in (M, g) , so by definition 1.3 the ADM mass of g is an upper bound for the outer mass of (S^2, γ, H) , whence by item (v) of the proposition

$$(3.1) \quad m_{out}[\gamma, H] \leq \frac{1}{16\pi} \int_{S^2} (6 + 2H - \gamma^\mu_\mu) + C \left(\|\gamma_{\mu\nu} - (\iota^*\delta)_{\mu\nu}\|_{3,\alpha} + \|H + 2\|_{2,\alpha} \right)^2.$$

In contrast to the elementary calculations leading to the upper bound, for the lower bound we appeal to the inverse mean curvature flow of Huisken and Ilmanen. It is a well-known consequence of their proof [17] of the Riemannian Penrose Inequality (see in particular the remarks on page 426 preceding the proof of Positivity Property 9.1) that the ADM mass of an asymptotically flat manifold satisfying the decay conditions 1.1 (or somewhat weaker conditions) and having nonnegative scalar curvature is no less than the Hawking mass m_H of any outer-minimizing sphere it contains. Thus

$$(3.2) \quad m_{out}[\gamma, H] \geq m_H[\gamma, H] := \sqrt{\frac{\int_{S^2} \sqrt{|\gamma|}}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{S^2} H^2 \sqrt{|\gamma|} \right),$$

where $\sqrt{|\gamma|}$ denotes the integration measure induced on S^2 by γ . On the other hand a quick calculation reveals that the linearization of m_H at $(\iota^*\delta, -2)$ is just the first term of 3.1, which completes the proof of the theorem.

Proof of Corollary 1.8. To prove the corollary let p be a point in a smooth Riemannian manifold (N, g) , and for each small $\tau > 0$ let B_τ be the geodesic ball of center p and radius τ . Setting $\gamma = \iota^*\tau^{-2}g$ and $H = \mathcal{H}_\iota[\tau^{-2}g]$ and writing R_{ab} for the Ricci curvature of g at p and x^i for the i^{th} Cartesian coordinate function on \mathbb{R}^3 restricted to S^2 , by Taylor expansion in τ of γ and H (included in Appendix C), we compute

$$(3.3) \quad \begin{aligned} \frac{1}{16\pi} \int_{S^2} (6 + 2H - \gamma^\mu_\mu) &= \frac{1}{16\pi} \int_{S^2} \left(\tau^2 R_{ij} x^i x^j + \frac{2}{3} \tau^3 R_{ij|k} x^i x^j x^k + \frac{1}{4} \tau^4 R_{ij|kl} x^i x^j x^k x^l \right) + O(\tau^5) \\ &= \frac{1}{48\pi} \tau^2 R_{ij} \int_{S^2} |x|^2 \delta^{ij} + 0 + \frac{1}{64\pi} \tau^4 R_{ij|kl} \int_B \partial^i (x^j x^k x^l) + O(\tau^5) \\ &= \frac{1}{12} R \tau^2 + \frac{1}{192\pi} \tau^4 R_{ij|kl} \int_B |x|^2 (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) + O(\tau^5) \\ &= \frac{1}{12} R \tau^2 + \frac{1}{120} \Delta R \tau^4 + O(\tau^5) \end{aligned}$$

and

$$(3.4) \quad \|\gamma - \iota^*\delta\|_{C^4}^2 + \|H + 2\|_{C^3}^2 \leq R_{ijkl} R^{ijkl} \tau^4 + O(\tau^5).$$

It follows from the theorem that

$$(3.5) \quad m_{out}[B_\tau, \tau^{-2}g] = \begin{cases} \frac{1}{12} R \tau^2 + O(\tau^4) & \text{in general} \\ \frac{1}{120} \Delta R \tau^4 + O(\tau^5) & \text{if } g \text{ is flat at } p. \end{cases}$$

Now the scaling law $m_{ADM}[M, \lambda^2 h] = \lambda m_{ADM}[M, h]$ for the ADM mass implies $m_{out}[B_\tau, g] = \tau m_{out}[B_\tau, \tau^{-2}g]$, completing the proof of the corollary.

APPENDIX A. DECAY ESTIMATE FOR THE NEWTON POTENTIAL

Let $\beta \in (0, \infty) \setminus \{1\}$ and $f \in C^{0,0,2+\beta}(\mathbb{R}^3)$. Then for $|x| \geq 1$

$$\begin{aligned}
 \|f\|_{0,0,2+\beta}^{-1} |(G * f)(x)| &\lesssim \int_{2|y| \leq |x|} |y|^{-1} (1 + |x - y|)^{-2-\beta} d^3y \\
 &\quad + \int_{|x| \leq 2|y| \leq 4|x|} |y|^{-1} (1 + |x - y|)^{-2-\beta} d^3y \\
 &\quad + \int_{|y| \geq 2|x|} |y|^{-1} (1 + |x - y|)^{-2-\beta} d^3y \\
 (A.1) \quad &\lesssim (1 + |x|)^{-2-\beta} \int_0^{|x|/2} r^{-1} r^2 dr + |x|^{-1} \int_0^{3|x|} (1 + r)^{-2-\beta} r^2 dr \\
 &\quad + \int_{2|x|}^\infty r^{-1} (1 + r)^{-2-\beta} r^2 dr \\
 &\lesssim (1 + |x|)^{-\beta} + (1 + |x|)^{-1} (1 - \beta)^{-1} \left((1 + |x|)^{1-\beta} - 1 \right) \\
 &\lesssim \begin{cases} (1 + |x|)^{-\beta} & \text{for } \beta < 1 \\ (1 + |x|)^{-1} & \text{for } \beta > 1. \end{cases}
 \end{aligned}$$

APPENDIX B. VARIATION OF THE BOUNDARY DATA WITH RESPECT TO AMBIENT METRIC

Let M be a manifold equipped with a smooth family $\{g_t\}_{t \in \mathbb{R}}$ of Riemannian metrics (with the t subscript frequently suppressed below), and let $\phi : \Sigma \rightarrow M$ be a codimension-one immersion, which we may assume to be a two-sided embedding, since the following calculations are purely local. We indicate t -derivatives by overhead dots, and, at any given value of t , indices are raised and lowered by g_t . Fix a (t -dependent) unit normal N^a and write ν^a for its value at $t = 0$. An n index on a tensor will indicate contraction with N . Since the 1-form N_a is always proportional to ν_a and since

$$(B.1) \quad 0 = \frac{d}{dt}(g^{ab} N_a N_b) = -\dot{g}^{nn} + 2\dot{N}_a N^a,$$

we have

$$(B.2) \quad \dot{N}_a = \frac{1}{2} \dot{g}^{bc} N_a N_b N_c = \frac{1}{2} \dot{g}^{nn} N_a$$

We write $A_{\alpha\beta}$ for the (t -dependent) scalar-valued second fundamental form of ϕ relative to g_{ab} and N^a . We extend N^a (a section of $\phi^*(TM)$) by (t -dependent) parallel transport along the (t -dependent) geodesics it generates to a vector field (a section of TM) of the same name on a neighborhood of Σ . Then

$$(B.3) \quad A_{\alpha\beta} = -\phi^a_{,\alpha} \phi^b_{,\beta} N_{a|b},$$

using $|$ indicate covariant differentiation defined by g_t . Where convenient we will alternatively use \overline{D} to represent the same Levi-Civita connection induced by g_t , and we define \overline{D}_{ab}^c by $\frac{d}{dt} \overline{D}_X Y = \overline{D}(X, Y)$ for X^a, Y^a independent of t . Now we compute

$$\begin{aligned}
 (B.4) \quad \frac{d}{dt} N_{a|b} &= -\dot{\overline{D}}_{abc} N^c + \overline{D}_b \dot{N}_a \\
 &= -\frac{1}{2} (\dot{g}_{ac|b} + \dot{g}_{bc|a} - \dot{g}_{ab|c}) N^c + \frac{1}{2} \dot{g}^{nn} N_{a|b} + \frac{1}{2} \dot{g}^{nn}_{,b} N_a,
 \end{aligned}$$

so

$$(B.5) \quad \dot{A}_{\alpha\beta} = \frac{1}{2} A_{\alpha\beta} \dot{g}^{nn} + \frac{1}{2} \phi^a_{,\alpha} \phi^b_{,\beta} (\dot{g}_{an|b} + \dot{g}_{bn|a} - \dot{g}_{ab|n})$$

and therefore, writing $H := (\phi^*g)^{\alpha\beta} A_{\alpha\beta}$ for the scalar-valued mean curvature of ϕ relative to g_t and N_t (and raising Greek indices via ϕ^*g_t),

$$(B.6) \quad \dot{H} = -\phi^a{}_{,\alpha}\phi^b{}_{,\beta}\dot{g}_{ab}A^{\alpha\beta} + \frac{1}{2}H\dot{g}^{nn} + \frac{1}{2}\phi^a{}_{,\alpha}\phi^{b,\alpha}(\dot{g}_{an|b} + \dot{g}_{bn|a} - \dot{g}_{ab|n}).$$

Diffeomorphisms. Now suppose g_0 is a given metric on M and suppose ν^a is a unit normal as above, extended, without relabelling, to a vector field on a neighborhood of Σ by parallel translation along the geodesics it generates. We consider g_t defined by the evolution of g_0 under the flow generated by a t -independent vector field ξ^a .

Normal fields. First suppose $\xi^a = f\nu^a$ for some t -independent function $f \in C_{loc}^2(M)$. Then

$$(B.7) \quad \dot{g}_{ab} = f|_b\nu_a + f|_a\nu_b + 2f\nu_a|_b,$$

so

$$(B.8) \quad \frac{d}{dt}(\phi^*g)_{\alpha\beta} = -2A_{\alpha\beta}\phi^*f,$$

$$(B.9) \quad \dot{g}^{nn} = 2f_{,n}$$

and

$$(B.10) \quad \dot{g}_{ab|c} = f|_{bc}\nu_a + f|_b\nu_{a|c} + f|_{ac}\nu_b + f|_a\nu_{b|c} + 2f|_c\nu_a|_b + 2f\nu_a|_{bc},$$

whence

$$(B.11) \quad \begin{aligned} \phi^a{}_{,\alpha}\phi^b{}_{,\beta}\dot{g}_{ab|n} &= -2(\phi^*f_{,n})A_{\alpha\beta} + 2\phi^a{}_{,\alpha}\phi^b{}_{,\beta}\nu_a|_{bn} \\ &= -2(\phi^*f_{,n})A_{\alpha\beta} + 2\phi^a{}_{,\alpha}\phi^b{}_{,\beta}\nu_a|_{nb} + 2\phi^a{}_{,\alpha}\phi^b{}_{,\beta}R_{nbna}, \end{aligned}$$

with curvature convention $R_{abcd} = \langle \overline{D}_{\partial_a}\overline{D}_{\partial_b}\partial_c - \overline{D}_{\partial_b}\overline{D}_{\partial_a}\partial_c, \partial_d \rangle$, and

$$(B.12) \quad \phi^a{}_{,\alpha}\phi^b{}_{,\beta}\dot{g}_{an|b} = -(\phi^*f_{,n})A_{\alpha\beta} + \phi^a{}_{,\alpha}\phi^b{}_{,\beta}f|_{ab} + 2\phi^a{}_{,\alpha}\phi^b{}_{,\beta}f\nu_a|_{nb},$$

so using also

$$(B.13) \quad \phi^a{}_{,\alpha}\phi^b{}_{,\beta}f|_{ab} = (\phi^*f)_{;\alpha\beta} - A_{\alpha\beta}\phi^*f_{,\nu},$$

with $:$ indicating covariant differentiation relative to ϕ^*g , and

$$(B.14) \quad \nu_a|_{nb} = \nu_n|_{ab} = -\nu_a|_c\nu^c{}_b$$

we obtain from B.6

$$(B.15) \quad \dot{H} = (\phi^*f)_{;\alpha}{}^{;\alpha} + |A|^2\phi^*f + \phi^*R_{nn}f.$$

Tangential fields. Now suppose $\xi^a = W^a$ for $W \in C_{loc}^2(TM)$ everywhere orthogonal to ν . Then W restricts to a vector field on Σ which we will also call W . Then

$$(B.16) \quad \dot{g}_{ab} = W_a|_b + W_b|_a,$$

so

$$(B.17) \quad \begin{aligned} \frac{d}{dt}(\phi^*g)_{\alpha\beta} &= W_{\alpha;\beta} + W_{\beta;\alpha}, \\ \dot{g}^{nn} &= 0, \\ \phi^a{}_{,\alpha}\phi^b{}_{,\beta}\dot{g}_{ab|n} &= \phi^a{}_{,\alpha}\phi^b{}_{,\beta}(W_a|_{bn} + W_b|_{an}), \text{ and} \\ \phi^a{}_{,\alpha}\phi^b{}_{,\beta}\dot{g}_{an|b} &= \phi^a{}_{,\alpha}\phi^b{}_{,\beta}(W_a|_{nb} + W_n|_{ab}), \end{aligned}$$

whereby

$$(B.18) \quad \begin{aligned} \frac{1}{2} \phi^a{}_{,\alpha} \phi^{b,\alpha} (\dot{g}_{an|b} + \dot{g}_{bn|a} - \dot{g}_{ab|n}) &= \phi^a{}_{,\alpha} \phi^{b,\alpha} (W_{a|nb} + W_{n|ab} - W_{a|bn}) \\ &= \phi^a{}_{,\alpha} \phi^{b,\alpha} W_{n|ab} + R_{nc} W^c, \end{aligned}$$

but the first term in B.6 is

$$(B.19) \quad \begin{aligned} -\phi^a{}_{,\alpha} \phi^{b,\alpha} \dot{g}_{ab} A^{\alpha\beta} &= 2W_{a|b} N^{a|b} \\ &= (W_a N^a)^b{}_{|b} - W_{a|b}{}^b N^a - W_a N^{a|b}{}_{|b} \\ &= 0 - W_{n|c}{}^c - W^a N^b{}_{|ab} \\ &= -W_{n|c}{}^c - R_{an} W^a + W^c H_{|c}, \end{aligned}$$

and therefore, noting $W_{n|n}{}^n = 0$,

$$(B.20) \quad \dot{H} = WH.$$

Conformal change. Of course we can also apply B.6 to linearized conformal transformations, but the noninfinitesimal transformation laws are simple enough. If h_{ab} is a fixed metric on M and $\rho \in C^1_{loc}(M)$ is everywhere strictly positive, then under the conformal change $\tilde{h}_{ab} = \rho^2 h_{ab}$ we have the corresponding conformal change $\phi^* \tilde{h} = (\phi^* \rho^2)(\phi^* h)$ for the induced metric on Σ . For the change of its second fundamental form from A to \tilde{A} , starting from the identity $A = -\frac{1}{2} \phi^* L_N h$ we compute

$$(B.21) \quad \begin{aligned} \tilde{A} &= -\frac{1}{2} \phi^* L_{\rho^{-1}N}(\rho^2 h) \\ &= -\frac{1}{2} (\phi^* \rho^{-1}) \phi^* L_N(\rho^2 h) \quad (\text{since } N \perp \phi_* T\Sigma) \\ &= (\phi^* \rho) A - (\phi^* \rho_{,n}) \phi^* h, \end{aligned}$$

whence

$$(B.22) \quad \tilde{H} = (\phi^* \rho^{-1}) H - (\dim \Sigma) (\phi^* \rho^{-2} \rho_{,n}).$$

Thus for a conformal family of metrics $g_t = \rho^2(t) g_0$ with $\rho(0) = 1$

$$(B.23) \quad \begin{aligned} \dot{H}_0 &= -(\phi^* \dot{\rho}) H_0 - (\dim \Sigma) (\phi^* \dot{\rho}_{,n}) \\ &= -\frac{H}{2} \frac{d}{dt} \Big|_{t=0} \phi^* \rho^2 - \frac{\dim \Sigma}{2} \frac{d}{dt} \Big|_{t=0} \phi^* (\rho^2)_{,n}. \end{aligned}$$

APPENDIX C. THE INDUCED METRIC OF SMALL METRIC SPHERES

Fix a point p in a Riemannian manifold (M, g) of dimension $n+1$, and let S^n be the unit n -sphere centered at the origin in $T_p M$. Define $\Phi : \mathbb{R} \times S^n \rightarrow M$ by

$$(C.1) \quad \Phi(t, \theta) = \exp_p t\theta,$$

where \exp is the exponential map corresponding to g . We will casually identify vector fields on S^n with $T_p M$ -valued maps on S^n (whose preimages are orthogonal to their images) as well as with their t -independent extensions to $\mathbb{R} \times S^n$. Writing \overline{D} for the Levi-Civita connection on TM determined by g , there is a unique connection D on $\Phi^* TM$ satisfying $D_X \Phi^* \xi = \overline{D}_{\Phi_* X} \xi$ for any $X \in C^0_{loc}(T(\mathbb{R} \times S^n))$, $\xi \in C^1_{loc}(TM)$; it is torsion-free in the sense that $D_X \Phi_* Y - D_Y \Phi_* X = \Phi_* [X, Y]$

and metric-compatible. Writing P_s^t for the corresponding parallel-transport map from the fiber over (s, \cdot) to the fiber over (t, \cdot) , define further the time-dependent map $\phi(t)$ from TS^n to $T_p M$ by

$$(C.2) \quad \phi(t)V = P_t^0 \Phi_* V.$$

Defining also the parallelly transported curvature operator \bar{R}^a_{rbr} by $\bar{R}^a_{rbr} V^b = P_t^0 R(\Phi_* \partial_t, V) \Phi_* \partial_t$ and with similar notation for covariant derivatives of curvature, we have

$$(C.3) \quad \begin{aligned} \phi(0) &= 0, \\ \dot{\phi}(0)V &= D_V \Phi_* \partial_t|_{t=0} = V, \\ \ddot{\phi}^a_b(t) &= \bar{R}^a_{rcr} \phi^c_b, \\ \dddot{\phi}^a_b(t) &= \bar{R}^a_{rcr|r} \phi^c_b + \bar{R}^a_{rcr} \dot{\phi}^c_b, \\ \ddot{\phi}^a_b(t) &= \bar{R}^a_{rcr|rr} \phi^c_b + 2\bar{R}^a_{rcr|r} \dot{\phi}^c_b + \bar{R}^a_{rcr} \ddot{\phi}^c_b, \\ \ddot{\phi}^a_b(t) &= \bar{R}^a_{rcr|rrr} \phi^c_b + 3\bar{R}^a_{rcr|rr} \dot{\phi}^c_b + 3\bar{R}^a_{rcr|r} \ddot{\phi}^c_b + \bar{R}^a_{rcr} \ddot{\phi}^c_b, \text{ and} \\ \ddot{\phi}^a_b(t) &= \bar{R}^a_{rc|rrrr} \phi^c_b + 4\bar{R}^a_{rcr|rrr} \dot{\phi}^c_b + 6\bar{R}^a_{rcr|rr} \ddot{\phi}^c_b + 4\bar{R}^a_{rcr|r} \ddot{\phi}^c_b + \bar{R}^a_{rcr} \ddot{\phi}^c_b, \end{aligned}$$

so

$$(C.4) \quad \begin{aligned} \phi^a_b(t) &= t\delta^a_b + \frac{1}{6}t^3 \bar{R}^a_{rbr}(0) + \frac{1}{12}t^4 \bar{R}^a_{rbr|r}(0) + \frac{1}{40}t^5 \bar{R}^a_{rbr|rr}(0) + \frac{1}{120}t^5 R^a_{rcr} \bar{R}^c_{rbr}(0) \\ &\quad - \frac{1}{5!} \int_0^1 (t-1)^5 \ddot{\phi}^a_b(t) dt. \end{aligned}$$

Write $\gamma(t)$, $A(t)$, and $H(t)$ for the metric, second fundamental form, and mean curvature induced by g on the metric sphere of radius t . Since $\gamma_t(V, W) = g_p(\phi V, \phi W)$, for small t we obtain

$$(C.5) \quad \left\| t^{-2} \gamma_{\mu\nu}(t) - \left[\iota^* \delta_{\mu\nu} + \frac{1}{3}t^2 R_{\mu r \nu r} + \frac{1}{6}t^3 R_{\mu r \nu r|r} + t^4 \left(\frac{1}{20} R_{\mu r \nu r|rr} + \frac{2}{45} R_{\mu r \lambda r} R^\lambda_{r \nu r} \right) \right] \right\|_{C^k} \leq C(k)t^5,$$

$$(C.6) \quad \left\| t^2 \gamma^{\mu\nu}(t) - \left[\iota^* \delta^{\mu\nu} - \frac{1}{3}t^2 R^\mu_r{}^\nu_r - \frac{1}{6}t^3 R^\mu_r{}^\nu_{r|r} - t^4 \left(\frac{1}{20} R^\mu_r{}^\nu_{r|rr} - \frac{1}{15} R^\mu_{r \lambda r} R^\lambda_r{}^\nu_r \right) \right] \right\|_{C^k} \leq C(k)t^5,$$

$$(C.7) \quad \left\| t^{-1} A(t) - \left[-\iota^* \delta_{\mu\nu} - \frac{2}{3}t^2 R_{\mu r \nu r} - \frac{5}{12}t^3 R_{\mu r \nu r|r} - t^4 \left(\frac{3}{20} R_{\mu r \nu r|rr} + \frac{2}{15} R_{\mu r \lambda r} R^\lambda_{r \nu r} \right) \right] \right\|_{C^k} \leq C(k)t^5, \text{ and}$$

$$(C.8) \quad \left\| tH(t) - \left[-2 + \frac{1}{3}t^2 R_{rr} + \frac{1}{4}t^3 R_{rr|r} + t^4 \left(\frac{1}{10} R_{rr|rr} + \frac{1}{45} R_{\mu r \nu r} R^{\mu r \nu r} \right) \right] \right\|_{C^k} \leq C(k)t^5,$$

where the curvature factors are all evaluated at p (and we observe the same index-ordering convention as for \bar{R} above).

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